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Interface Sharpness in the Potts Model

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A simple proof is given for the existence of a sharp interface between two ordered phases for the three-dimensional 2^n-state Potts model (n integer).

KEY WORDS: Potts model; interfaces.

1. INTRODUCTION

Among the models which exhibit phase transitions is the well-known Ising model, for which the properties of the interface between the + and − phases coexisting at low temperature have been widely studied. In particular, Dobrushin[(1)] has shown that in three dimensions this interface is sharp at low temperature. By a different and very simple method van Beijeren[(2)] has shown that this interface is sharp for every temperature smaller than the critical temperature of the two-dimensional Ising model.

Richer possibilities in such problems occur when three or more phases coexist. This is the case for the q-state Potts model introduced in 1952[(3)] as a generalization of the Ising model by enlarging the values that the spins can take from two to an arbitrary integer q. As 1 shall recall more precisely in Section 2, in two or more dimensions, this model exhibits, for q large enough, a first-order phase transition at some inverse temperature β_t where q ordered phases coexist with a disordered one. Above β_t only ordered phases coexist and below β_t there is a unique phase; the surface tensions between two ordered phases and the surface tensions betweens any ordered and the disordered phase are strictly positive when the considered phases coexist.

I here prove that in three dimensions the interface between two ordered phases is sharp, if $q = 2^n$ ($n \geq 1$), for any temperature smaller than

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the transition temperature of the two-dimensional q-state Potts model. I follow the method of van Beijeren using the spin-1/2 representation of the Potts model,[4] which allows use of the Lebowitz inequality.[5] The use of the FKG inequality[6] in the van Beijeren method will be replaced in our case by a correlation inequality implying in particular that at β_t the interface between two ordered phases is wetted by a film of the disordered phase.[7]

The paper is organized as follows. In Section 2, I recall the phase diagram of the Potts model and show that several definitions of different authors for the disordered phase coincide. The main results are given in Section 3 and concluding remarks in Section 4.

2. THE POTTS MODEL AND ITS PHASE DIAGRAM

To introduce the q-state Potts model, associate at each lattice site $i \in \mathbb{Z}^d$ a spin x_i taking its values in the set $\{1,..., q\}$. The Hamiltonian in a finite box $\Lambda \subset \mathbb{Z}^d$ is given by

$$\mathscr{H}_\Lambda = - \sum_{\langle i;j \rangle} \delta(x_i, x_j) \tag{1}$$

where the bracket $\langle\, ; \rangle$ restricts the sum over nearest neighbor (n.n.) pairs and δ is the Kronecker symbol. Denote by $\langle \cdot \rangle^{\mathrm{f}}_\Lambda(\beta)$ the expectation values corresponding to the Gibbs measure $Z_\Lambda^{-1} \exp\{-\beta \mathscr{H}_\Lambda\}$, and by $\langle \cdot \rangle^{\mathrm{f}}(\beta)$ its infinite-volume limit obtained by letting $\Lambda \uparrow \mathbb{Z}^d$. They corresponding to the so-called free (f) boundary condition (b.c.). We also introduce other b.c. with the use of an infinite external field. Agreement of the notation with Section 3 is the reason for doing this in such a way. Namely, we add to $\mathscr{H}_\Lambda$ the term

$$\mathscr{B}_\Lambda = - \sum_{i \in \Lambda} H_i \delta(x_i, a) - \sum_{i \in \Lambda} K_i \delta(x_i, b) \tag{2}$$

where all the H_i and K_i are nonnegative, and $a, b \in \{1,..., q\}$. The so-called closed or ordered b.c. corresponds to infinite H_i for every site i in the boundary $\partial\Lambda$ of Λ ($\partial\Lambda = \{i \in \Lambda / \exists j \in \mathbb{Z}^d \setminus \Lambda$ s.t. i and j are n.n.$\}$), while H_i is zero otherwise and K_i is zero for every i in Λ.

Whenever q is large enough, the phase diagram of the d-dimensional ($d \geqslant 2$) q-state Potts model is as follows.

There exists a unique inverse temperature β_t, where the magnetization $M_{d,q} = (q-1)^{-1} \langle q\delta(x_i, a) - 1 \rangle^a(\beta)$ is discontinuous, such that:

(a) For $\beta < \beta_t$ there is a unique phase.

(b) For $\beta > \beta_t$ every translation-invariant (TI) Gibbs state $\langle \cdot \rangle^{\mathrm{I}}(\beta)$ is a convex linear combination of the q ordered states:

$$\langle \cdot \rangle^{\mathrm{I}}((\beta) = \sum_{n=1}^{q} \lambda_n \langle \cdot \rangle^n (\beta), \qquad \sum_{n=1}^{q} \lambda_n = 1$$

(c) For $\beta = \beta_t$ every TI state is expressible as

$$\langle \cdot \rangle^{\mathrm{I}}(\beta) = \sum_{n=1}^{q} \lambda_n \langle \cdot \rangle^n (\beta) + \lambda_{\mathrm{f}} \langle \cdot \rangle^{\mathrm{f}}(\beta), \qquad \sum_{n=1}^{q} \lambda_n + \lambda_{\mathrm{f}} = 1$$

Statements (a)–(c) have been proved by Martirosian,[8] but with a slight difference. Namely, in ref. 8, statement (c) is

$$\langle \cdot \rangle^{\mathrm{I}}(\beta_t) = \sum_{n=1}^{q} \lambda'_n \langle \cdot \rangle^n (\beta_t) + \lambda'_{\mathrm{dis}} \langle \cdot \rangle^{\mathrm{dis}}(\beta_t) \tag{3}$$

where the state $\langle \cdot \rangle^{\mathrm{dis}}$ is obtained with the so-called disordered b.c. Statement (c) can be proved in the following way.

Proof of Statement (c). The free state is translation invariant: the proof is standard[9] once one has monotonicity properties in the volume; in our case they follow from the Ginibre inequality,[10] which applies to the Potts model, as noticed in ref. 11. Starting from (3), we have

$$\langle \cdot \rangle^{\mathrm{f}}(\beta_t) = \sum_{n=1}^{q} \lambda'_n \langle \cdot \rangle^n (\beta_t) + \lambda'_{\mathrm{dis}} \langle \cdot \rangle^{\mathrm{dis}}(\beta_t) \tag{4}$$

In particular, this implies for any n.n. pair i, j

$$\sum_{n=1}^{q} \lambda'_n [\langle \delta(x_i, x_j) \rangle^n (\beta_t) - \langle \delta(x_i, x_j) \rangle^{\mathrm{f}} (\beta_t)] \\ + \lambda'_{\mathrm{dis}} [\langle \delta(x_i, x_j) \rangle^{\mathrm{dis}} - \langle \delta(x_i, x_j) \rangle^{\mathrm{f}} (\beta_t)] = 0 \tag{5}$$

We then use that for any TI state and any $n \in \{1,..., q\}$

$$\langle \delta(x_i, x_j) \rangle^{\mathrm{f}} (\beta) \leqslant \langle \delta(x_i, x_j) \rangle^{\mathrm{I}} (\beta) \leqslant \langle \delta(x_i, x_j) \rangle^n (\beta) \tag{6}$$

as shown by Pfister[12]; on the other hand, there exists some β'_t such that

$$\langle \delta(x_i, x_j) \rangle^n (\beta'_t) > \langle \delta(x_i, x_j) \rangle^{\mathrm{f}} (\beta'_t) \tag{7}$$

as shown in refs. 13 and 14. Statements (a) and (b) imply that $\beta'_t = \beta_t$ and then we immediately get that relations (5)–(7) imply $\lambda'_n = 0$ for every n and hence $\lambda'_{\text{dis}} = 1$ since $\sum_n \lambda'_n + \lambda'_{\text{dis}} = 1$. Therefore the free state coincides with the disordered one. ∎

Note that they therefore coincide also with the state $\langle \cdot \rangle^{\neq}$ defined by Kotecky and Shlosman,[15] as conjectured by Pfister.[12]

In two dimensions Potts[3] conjectured the transition point to be $\beta_t(q, 2) = \log(\sqrt{q} + 1)$, the self-dual point. Baxter[17] has shown that a first-order transition indeed occurs at the conjectured point, for all $q \geqslant 4$, with a jump of magnetization; for $q \leqslant 4$ he got a continuous transition. Hintermann *et al.*[16] proved the uniqueness of the transition for $q > 4$, showing in particular that the free energy is analytic out of the self-dual point. Thus, for $q \geqslant 4$ and $d = 2$, this result and the analysis of TI states of ferromagnetic systems in ref. 12 imply the statements (a) and (b), i.e., the phase diagram out of the transition point. I now turn to the main object of this paper.

3. MAIN RESULT

I first introduce the spin-1/2 representation.[4] Whenever $q = 2^n$, n integer, the states x_i can be put in one-to-one correspondence with a configuration of n spins $\sigma_i^\alpha = \pm 1$, $\alpha = 1,..., n$, which can be thought of as all being at the same site i or as living on n copies of the lattice. We have

$$\delta(x_i, x_j) = \prod_{\alpha=1}^{n} \frac{1 + \sigma_i^\alpha \sigma_j^\alpha}{2} = 2^{-n} \sum_E \prod_{\alpha \in E} \sigma_i^\alpha \sigma_j^\alpha + 2^{-n} \tag{8}$$

where the sum runs over all nonempty subsets E of $\{1,..., n\}$ and if a is the particle $(1,..., 1)$,

$$\delta(x_i, a) = 2^{-n} \sum_E \prod_{\alpha \in E} \sigma_i^\alpha + 2^{-n} \tag{9}$$

I now apply the method of van Beijeren to this system. To this end, consider a simple cubic lattice Λ on a cube of $2N+1$ horizontal layers numbered $-N$, $-N+1,..., N$. The spins on the central layer 0 are numbered σ_m^α, σ_n^α,..., $\alpha = 1,..., n$, the spins on the layers $1,..., N$ are numbered $\sigma_i^\alpha, \sigma_j^\alpha$,..., and those on the layers $-1,..., -N$ are numbered $\sigma_{-i}^\alpha, \sigma_{-j}^\alpha$,.... This numbering is chosen such that the sites i and $-i$ are mirror images of each other with respect to the central layer. Also consider a two-dimensional square lattice Ω of $(2N+1) \times (2N+1)$ sites with spins numbered $\bar{\sigma}_m^\alpha$, $\bar{\sigma}_n^\alpha$,... (all the spins in both systems may assume the values ± 1 and $\alpha = 1,..., n$). The Hamiltonians of both systems are given by

$$H_\Lambda(\sigma)=J\left[\sum_{\langle i,j\rangle}\left(\prod_{\alpha=1}^{n}\frac{1+\sigma_i^\alpha\sigma_j^\alpha}{2}+\prod_{\alpha=1}^{n}\frac{1+\sigma_{-i}^\alpha\sigma_{-j}^\alpha}{2}\right)+\sum_{\langle m,n\rangle}\prod_{\alpha=1}^{n}\frac{1+\sigma_m^\alpha\sigma_n^\alpha}{2}\right]$$
$$+J\sum_{\langle i,m\rangle}\left(\prod_{\alpha=1}^{n}\frac{1+\sigma_m^\alpha\sigma_i^\alpha}{2}+\prod_{\alpha=1}^{n}\frac{1+\sigma_m^\alpha\sigma_{-i}^\alpha}{2}\right)$$
$$+\sum_i h_i\prod_{\alpha=1}^{n}\frac{1+\sigma_i^\alpha}{2}+\sum_i h_i\frac{1-\sigma_{-i}^1}{2}\prod_{\alpha=2}^{n}\frac{1+\sigma_{-i}^\alpha}{2}$$
$$+\sum_m H_m\prod_{\alpha=1}^{n}\frac{1+\sigma_m^\alpha}{2} \tag{10}$$

$$H_\Omega(\bar\sigma)=\sum_{\langle m,n\rangle}\prod_{\alpha=1}^{n}\frac{1+\bar\sigma_m^\alpha\bar\sigma_n^\alpha}{2}+\sum_m H_m\prod_{\alpha=1}^{n}\frac{1+\bar\sigma_m^\alpha}{2} \tag{11}$$

All the h_i and H_m are nonnegative. We are especially interested in the case where h_i and H_m are $+\infty$ at the boundary sites and zero otherwise. This gives for Λ a system with mixed boundary conditions $a\equiv(1, 1,..., 1)$ on the top half and $b\equiv(-1, 1,..., 1)$ on the bottom half.[2]

I now define, in analogy with the method of Percus,[18] the new variables

$$s_i^\alpha=\tfrac{1}{2}(\sigma_i^\alpha+\sigma_{-i}^\alpha),\qquad s_m^\alpha=\tfrac{1}{2}(\sigma_m^\alpha+\bar\sigma_m^\alpha)$$
$$t_i^\alpha=\tfrac{1}{2}(\sigma_i^\alpha-\sigma_{-i}^\alpha),\qquad t_m^\alpha=\tfrac{1}{2}(\sigma_m^\alpha-\bar\sigma_m^\alpha)$$

which may assume the values $-1, 0, 1$ with the constraints

$$s_i^\alpha=0\rightarrow t_i^\alpha=\pm 1\qquad\text{and}\qquad s_i^\alpha=\pm\rightarrow t_i^\alpha=0$$

The sum of $H_\Lambda(\sigma)$ and $H_\Omega(\bar\sigma)$ can be expressed in these new variable as

$$H_\Lambda(\sigma)+H_\Omega(\bar\sigma)$$
$$=J'\sum_{\langle i,j\rangle}\sum_E\left[\prod_{\alpha\in E}(s_i^\alpha+t_i^\alpha)(s_j^\alpha+t_j^\alpha)+\prod_{\alpha\in E}(s_i^\alpha-s_i^\alpha)(s_j^\alpha-t_j^\alpha)\right]$$
$$+J'\sum_{\langle m,n\rangle}\sum_E\left[\prod_{\alpha\in E}(s_m^\alpha+t_m^\alpha)(s_n^\alpha+t_j^\alpha)+\prod_{\alpha\in E}(s_m^\alpha-s_m^\alpha)(s_n^\alpha-t_n^\alpha)\right]$$
$$+J'\sum_{\langle m,i\rangle}\sum_E\prod_{\alpha\in E}(s_m^\alpha+t_m^\alpha)\left[\prod_{\alpha\in E}(s_i^\alpha+t_i^\alpha)+\prod_{\alpha\in E}(s_i^\alpha-t_i^\alpha)\right]$$
$$+J'\sum_i h_i\sum_E\left[\prod_{\alpha\in E}(s_i^\alpha+t_i^\alpha)+\varepsilon(E)\prod_{\alpha\in E}(s_i^\alpha-t_i^\alpha)\right]$$
$$+J'\sum_m H_m\sum_E\left[\prod_{\alpha\in E}(s_m^\alpha+t_m^\alpha)+\prod_{\alpha\in E}(s_m^\alpha-t_m^\alpha)\right]$$
$$+2^{-n}[L+S-(2N+1)^2] \tag{12}$$

[2] In fact, one can choose for b any particle different from a.

where $J' = J/2^n$ and $\varepsilon(E)$ equals -1 if E contains 1, and 1 otherwise. L is the number of n.n. in Λ and S the number of sites in Λ.

Since the Hamiltonian (12) is clearly ferromagnetic in the s and t variables, the inequality,

$$\left\langle \prod_{\alpha \in E} s_m^\alpha \prod_{\alpha \in E'} t_m^\alpha \right\rangle \geqslant 0 \tag{13}$$

holds true for any $E, E' \subset \{1,..., n\}$. Here $\langle \cdot \rangle$ denotes the expectation with respect to the product measure of the canonical measure for σ and $\bar{\sigma}$, respectively. We deduce from (13) that for any subset E of $\{1,..., n\}$

$$\left\langle \prod_{\alpha \in E} \sigma_m^\alpha \right\rangle \geqslant \left\langle \prod_{\alpha \in E} \bar{\sigma}_m^\alpha \right\rangle \tag{14}$$

Denoting by $\langle \cdot \rangle_\Lambda^{ab}(J)$ the expectation for the canonical measure σ (with the above considered values of h_i and H_m), we get that the expectation value of

$$q\delta(x_m, a) - 1 = \sum_E \prod_{\alpha \in E} \sigma_m^\alpha \tag{15}$$

is greater than the spontaneous magnetization of the two-dimensional Potts model:

$$\langle q\delta(x_m, a) - 1 \rangle_\Lambda^{ab}(J) \geqslant \langle q\delta(\bar{x}_m, a) - 1 \rangle_\Omega^a(J) \geqslant (q-1)\, M_{2,q}(J) \tag{16}$$

and

$$\langle \delta(x_m, a) - \delta(x_m, c) \rangle_\Lambda^{ab}(J) \geqslant \langle \delta(\bar{x}_m, a) - \delta(\bar{x}_m, c) \rangle_\Omega^a(J) \tag{17}$$

for any c different from a. By the correlation inequalities derived in ref. 7 (see Theorem 1), we know that the expectation of (15) with respect to the Gibbs measure $Z_\Lambda^{-1} \exp[-\beta(\mathscr{H}_\Lambda + \mathscr{B}_\Lambda)]$ is increasing by applying a field in the direction of the particle a and decreasing by applying a field in the direction of the particle b. Namely, if $E(f; g) = E(fg) - E(f)\, E(g)$ denotes the truncated expectation of f and g with respect to this measure, then for any subsets A and B of Λ one has

$$\begin{aligned} &E\left(\prod_{i \in A} \delta(x_i, a); \prod_{j \in B} \delta(x_j, a) \right) \geqslant 0 \\ &E\left(\prod_{i \in A} \delta(x_i, a); \prod_{j \in B} \delta(x_j, b) \right) \leqslant 0 \end{aligned} \tag{18}$$

Therefore,

$$\langle \delta(\bar{x}_m, c) \rangle_\Omega^a(J) \leqslant \langle \delta(\bar{x}_m, c) \rangle_\Omega^{\mathrm{f}}(J) = \frac{1}{q}$$

and thus the difference of the lhs of inequality (17) is also greater than the spontaneous magnetization of the two-dimensional Potts model,

$$q\langle \delta(x_m, a) - \delta(x_m, c) \rangle_\Lambda^{ab}(J) \geqslant (q-1)\, M_{2,q}(J) \tag{19}$$

I now consider, as in ref. 2, the system with $2N$ layers and similar b.c., where there is a symmetry between the upper and lower halves of the system. This last system can be obtained in the following way. Start with a $(2N+1)$-layer system with infinite boundary field h_i and apply an infinite field to all spins in the layer N. The resulting system of $2N$ layers, Λ', has the a boundary condition on the top half and the b boundary condition on the bottom half. We denote by $\langle \cdot \rangle_{\Lambda'}^{ab}$ the expectation associated with this system. The fact that the expectation of (15) as well as the expectation of $\delta(x_m, a) - \delta(x_m, b)$ for this system are greater than the corresponding expectation for the $(2N+1)$-layer system is a consequence of correlation inequalities (18). Thus,

$$\begin{aligned} \langle q\delta(x_m, a) - 1 \rangle_{\Lambda'}^{ab}(J) &\geqslant \langle q\delta(x_m, a) - 1 \rangle_\Lambda^{ab}(J) \\ &\geqslant (q-1)\, M_{2,q}(J) \end{aligned} \tag{20}$$

$$\begin{aligned} q\langle \delta(x_m, a) - \delta(x_m, b) \rangle_{\Lambda'}^{ab}(J) &\geqslant q\langle \delta(x_m, a) - \delta(x_m, b) \rangle_\Lambda^{ab}(J) \\ &\geqslant (q-1)\, M_{2,q}(J) \end{aligned} \tag{21}$$

Moreover, over can easily show, by using the Fortuin–Kasteleyn random cluster expansion,[18] that for any $c \in \{1,..., q\}$, different from a and b,

$$\langle q\delta(x_m, c) - 1 \rangle_{\Lambda'}^{ab}(J) \leqslant 0$$

and we get

$$q\langle \delta(x_m, a) - \delta(x_m, c) \rangle_{\Lambda'}^{ab}(J) \geqslant \langle q\delta(x_m, a) - 1 \rangle_{\Lambda'}^{ab}(J) \geqslant (q-1)\, M_{2,q}(J) \tag{22}$$

By symmetry we obtain for any site i of the -1 layer and any particle d different from the particle b

$$\langle q\delta(x_i, b) - 1 \rangle_{\Lambda'}^{ab}(J) \geqslant (q-1)\, M_{2,q}(J) \tag{23}$$

$$q\langle \delta(x_i, b) - \delta(x_i, d) \rangle_{\Lambda'}^{ab}(J) \geqslant (q-1)\, M_{2,q}(J) \tag{24}$$

Therefore we conclude from (20)–(24) that the state $\langle \cdot \rangle^{ab}$ is non translation invariant and that under the boundary condition ab there is a sharp interface between the layers 0 and -1 provided the spontaneous magnetization of the two-dimensional Potts model is strictly positive. This is the case if $q > 4$ for $J \geqslant \beta_t(q, 2)$, and if $q \leqslant 4$ for $J > \beta_t(q, 2)$.

We clearly also prove (20)–(22) for any site of the layers $1,..., N-1$, and (23), (24) for any site of the layers $-2,..., -N$. Let us notice that whenever q is large and $J \geqslant \log(\sqrt{q}+1)$,

$$\langle \delta(x_m, a) \rangle_{\Lambda'}^{ab}(J) \geqslant 1 - O\left(\frac{1}{q}\right), \qquad \langle \delta(x_i, b) \rangle_{\Lambda'}^{ab}(J) \geqslant 1 - O\left(\frac{1}{q}\right)$$

Note some simple generalizations:

1. The lattice need not be a cube. Arbitrary shape may be considered, provided the layers n and $-n$ are minor images of each other with respect to the central layer.

2. The restriction to nearest neighbors may be loosened. The inequality (14) still holds, while inequalities (18) can been generalized to the case of arbitrary pair (not necessarily n.n.) interactions.[20] (For the Ising model, $q = 2$, the extension to long-range interactions is given in the Appendix B of ref. 21 and was previously noticed in ref. 2.)

3. The result generalizes to an arbitrary dimension d greater than 3.

4. CONCLUDING REMARKS

The above results show that the roughening temperature of the three-dimensional Potts model is greater than the transition temperature of the two-dimensional Potts model. For q large, it is expected that this roughening temperature is equal to the transition temperature (of the 3D model), i.e., that the interface between two ordered phases is sharp up to the transition temperature, at which it is wetted by a film of the disordered phase. It is also expected that for q large the a–f interface (between an ordered and the disordered phase) is sharp at the transition temperature.

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